GREATER DEPTH

IN PRIMARY MATHEMATICS

ANDREW JEFFREY

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Cover design by Lucy Rycroft-Smith

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Foreword

I’m delighted to recommend this latest offering from Andrew. Those who have read and used his previous publications know that he writes from experience of working with children in a huge variety of settings, both in the UK and abroad. If you are a practising teacher you will reflect on what is and what isn’t possible within your classroom. In our high stakes system we can be forgiven for putting test scores pretty high on the agenda

Andrew introduces strategies which not only support that type of success but which build future success too, and enable *all* children to experience what it is to ‘think like a mathematician’. In his usual magical way he mixes common sense with practical ideas, adds a flavouring of research and a sprinkle of humour and stirs it in the pot of current context. Enjoy!

Lynne McClure
Director, Cambridge Mathematics

Introduction

In September 2014 a new National Curriculum came into force in England. Around about the same time, what were referred to as 'National Curriculum Levels' were discontinued. These levels had been an attempt to bring consistency to the discussion of what children could do, but they were too broad to be of any real use, and soon the notion of sub-levels became ubiquitous.

The problem was that everyone interpreted these differently. They were used in many cases to supposedly demonstrate progress, or worse to hold teachers to account, and for a while in many schools teaching became all about moving children up through these arbitrary sub-levels as fast as possible to 'show progress'.

For me, one of the saddest moments of my career was one March day in the UK when I was visiting a school on behalf of the local authority, and asked where I could most be of use. They asked if I would be happy to work in Year 6 for the morning with their "four bees."

Not the "*children we think are working at level 4B*" or even worse, the "*4B children*" but simply the "four Bs". Simply put, as a profession we had begun to think of children not as wonderful unique human beings on an exciting learning journey, but as data points on a dubious and, certainly as far as sub-levels were concerned, a largely mythical scale.

The reasons are obvious in hindsight. Initially, the DfE said that a level 4 would be an average, meaning that many pupils were not expected to reach it - but over time schools which took that at face value were penalised as the bar continued to be raised; unofficially, nobody was supposed to have too many children below the average; mathematical nonsense of course.

There was also a discrepancy between levels that infant schools reported and those that junior schools found to be the case. It isn't hard to see why this would be so in such a high-stakes environment, and it was for many a huge relief to see levels finally scrapped in favour of a simpler 'year-end objectives' system.

But not for everyone. Many schools felt that they could no longer show progress without the benefit of levels and there was a rush to fill the void left by the departing assessment. The DfE appeared to have little idea either; they simply invited schools to create their own systems. From then on we have struggled to find a common language to talk about assessment, and it is from this point that the language of 'greater depth' began to emerge.

Given that all objectives were now to be reached '*by the end of the year*', a new scale began to emerge based on year groups , in direct contradiction to what the National Curriculum stated. This meant that children were either 'below', 'at' or 'above' the expected standard for that year group. Yet perhaps this misses the point of what learning truly is.

However convenient it might be, learning does not take place in 60 minute episodes. New ideas do not form in children's minds in neatly parcelled, daily hours, any more than they do in ours. Moreover, it is almost impossible to 'see' learning- we can only really see performance.

Year-end objectives are precisely that - the key word that is missed out is ‘by’, yet too often children are expected to meet an objective in one session or series of consecutive lessons. For example, year 1 children are required to:

‘recognise, find and name a quarter as one of four equal parts of an object, shape or quantity.’ (DfE, Mathematics Programme of Study, 2013)

As any Year 1 teacher will tell you, this happens gradually ‘by’ the end of the year, and over repeated visits, and certainly not altogether. All of this suggests that the way we approach our curriculum needs a radical rethink. Inevitably some children will arrive at a deeper level of understanding faster than others, and this creates a challenge for the teacher. What should we do with the children who appear to have internalised the concept of a quarter? Give them harder quantities and ask them to find a quarter of them? Give them an amount and ask what it might be a quarter of? Move them onto something different? Ask them to build a shape which is one quarter blue? Ask them to help those children who have not got the concept?

There is a place for many of these options, but not having a clear unequivocal policy nationally has led us to a somewhat fragmented point in our educational journey as a nation.

In an attempt to answer this question, the government produced a guidance document in which it specifically uses the words ‘greater depth’. Currently, this only exists for Key Stage 1 and not Key Stage 2. This is probably to discourage schools from using the document other than it was intended - as an ‘end of key stage only’ summative assessment tool.

The main thrust of this book, however, is unapologetically around classroom practices rather than assessment. In part one we will consider four key principles of ‘deeper thinking’, while part two will contains ten powerful and proven classroom strategies which aim to ensure that *all* children are given the opportunity to think and develop a sense of the mathematics around them.

This book has been written in such a way that you can read it cover to cover or just dip into the sections that interest you. In either case, you should find its key message to be crystal clear:

There is NO SUCH THING as a maths gene. With the exception of children with specific neurological conditions, everyone can be a successful mathematician. If you share this belief, then greater depth for all should be your goal.

Part 1

Four Characteristics of Greater Depth

Deep Thought paused for a moment's reflection.
"Tricky," he said finally.
"But can you do it?"
Again, a significant pause.
"Yes," said Deep Thought, "I can do it."
"There is an answer?" said Fook with breathless excitement.
"Yes," said Deep Thought. "Life, the Universe, and Everything. There is an answer. But, I'll have to think about it."

The Hitchhiker's Guide to the Galaxy, Douglas Adams.

1. Curiosity

"I wonder and am willing and able to explore multiple approaches and outcomes, purely for the sake of exploring them."

In 1988 the IBM computer known as Deep Thought, named after its fictional namesake (above) gained fame by becoming the first computer ever to beat a chess grandmaster. How was this achieved?

It is tempting to say that it was coded by chess experts but this is not the case. It was coded by engineers who knew the rules of the game and may even have been very good players, but who were by no means at Grandmaster level.

Curiously, despite its image as a 'clever' game, the rules of chess are actually very simple. Each piece can move according to strict pre-determined rules, and the objective is very simple to understand. What makes chess so complex however is the multiple permutations that may occur- no two games are likely to be the same.

So how does a computer like that end up beating a grandmaster? The answer is very simple - it had a very good memory. It could look ahead, metaphorically, and consider literally millions of possible outcomes. The very best chess players can also do this to a lesser extent, but with sufficient memory, a computer will be able to do so without ever misremembering. Deep Thought for example was capable of remembering up to 500 million possible outcomes, up to ten or so moves ahead, and this was enough to give it superiority over other 'chess computers' of its era, as well as the vast majority of its human opponents.

In the world of mathematics, the ability to hold in memory a range of facts and procedures is also a big advantage. More precisely, the ability to recall them when required is what gives the brain such a headstart.

Deep Thought was essentially just very, very good at asking the simple question “**what if…?**” and then carrying out thought experiments to discover the answer (or answers). Although very simply stated, this is one of the most profound questions that anyone hoping to develop their mathematical muscles can ever ask. *What if* we cut a square into quarters? *What if* we found 100 times 8.8; would that help us find 99 times 8.8? What if x was equal to 7? *What if* there was a largest prime? (which led to the proof that there cannot be*). But there is still one crucial part of the equation missing - having analysed the multiple possibilities, the computer selected the one most likely to be fruitful. This is very different to the way we expect children to learn.

Yet if we accept that the purpose of learning mathematics is to become a successful problem-solver, does it not stand to reason that we need to be faced with problems with which we are *not* familiar? The introduction of the 2014 National Curriculum stated that children should be able to tackle both *routine and non-routine problems* - so what is a routine problem? Presumably a problem where procedural fluency is required, but little reasoning.

Albert Einstein once said that he had no special talent, only that he was passionately curious. While we can assume that he was being somewhat modest, he certainly had a point. Curiosity, or the art of wondering, is a great gift. Mathematics originally evolved because the ancient Greeks were curious about the earth, for example. In fact it is where we derive our modern word 'geometry'; literally from '*geo-metria'*, the measuring of the Earth.

It has always struck me as both delightful and fortunate that children do not need to be taught curiosity, at least not in its natural form. You only need to watch the way in which young children interact with their environment to realise the truth of this. When brought face to face with a new object they will use all their senses to feel it, stare at it, rattle it (*and more often than not also taste it, whether or not we would consider it appropriate!*).

Young children have few boundaries where curiosity is concerned. So what happens to that wonderful innate curiosity? How, where and why does it diminish? It is here that we must take a long look at what happens in our homes and our schools.Homes and schools are sufficiently similar in this regard that we may consider them as one. So what exactly is it about these environments that inadvertently threatens to stifle curiosity with such regularity?

The Problem

Having thought about this for several years I now believe that it is something we may call the 'paradox of desirable conformity', which leads to the gradual blurring of rules and preferences.

By way of example, let me refer to my own childhood. I was brought up in quite a strict household, my father being a serving officer in the Royal Air Force, but it didn't feel untypical by the standards of the day. Like any family we had rules. I was not allowed to go near the cooker when my mother was boiling vegetables in hot saucepans. Nor was I allowed to rest my elbows on the table at mealtimes. Neither of these rules were unusual, and many of my friends had similar restrictions placed upon them.

Yet these two rules are very, very different from each other in one vital way - the first was designed entirely for my own safety. The second was nothing to do with my safety; it was a social convention, to do with perceived 'good manners' no more and no less. Nothing bad ever happened to me (other than the occasional reminder or telling off) whenever I inadvertently or even defiantly put my elbows on the table, nor was it likely to. It was a cultural rule rather than one designed to help me.

I saw this pattern repeated with my own children as they were growing up; we had and still have rules designed for their own safety, as well as others which are not to do with safety but which aim to help the family function as smoothly as possible. Some rules are designed in the interests of 'fairness'. But ultimately of course fairness is a subjective stance; *I* may think it is fair to require my teenagers to put the bins out, or to load the dishwasher, but *they* may not always see it that way.

What does all this have to do with curiosity? Discipline takes many forms, and since when we are young we typically have little self-discipline, and go wherever our innate curiosity leads, it is incumbent upon our schools and homes to decide on appropriate levels of discipline to impose upon children.

This discipline will inevitably lead to conflict - this is not a parenting manual, but let us consider just one example. The 'terrible twos', as they are colloquially known, is a well-known phenomenon where the imposition of discipline upon children comes into conflict with the child's growing independence and natural desire to behave differently. My will may be to follow my natural curiosity and climb into the frozen food section of the supermarket to see what it's like in there, and I will not take kindly to an adult preventing me from doing that. I am therefore likely to express my frustration and extreme displeasure at this unwarranted curbing of my curiosity in as loud and violent a manner as my two-year old lungs will permit.

We all know the scenario. But does this happen at school as well? Absolutely. Many of the rules that we impose on children are for their own safety. Don't run in the dining hall, don't go out of the school gate, don't climb the tree that is over the concrete, always sign in and out, don't leave the staffroom with un-lidded hot drinks, and so on.

A second category of school rules are those designed to support what we think of as the orderly running of the school. This is not to say that they are bad or wrong, of course. Rules encouraging us to speak kindly, to share, to develop empathy and so on, are things that focus on our values. It is frequently claimed that we are the product of our environment, and this includes not only our physical environment but also our social ethical and moral environments as well.

How does any of this impact upon our curiosity? In a nutshell, the more we are concentrating on and conforming to social norms, such as rules or expectations, the less time we have to wonder and discover. This is not to say that such rules and expectations are wrong of course - as families and schools, we have a natural and healthy desire to help shape the lives of the children in our care, but there are obvious and inherent dangers for curiosity; it becomes less and less valued on the altar of the expectations of others.

For example, I may wish to see what colourful patterns I can make - a white classroom wall, some paint and my hand would seem to fit the bill nicely, and I may be quite surprised when an adult appears to respond negatively to my beautiful mural. Readers who are lucky enough to work with children in the Early Years will doubtless be able to think of other examples of 'creativity' that do not sit well with the established expectations of the setting.

From my own experience, as a child with good analytical and reasoning skills yet significant (and at the time undiagnosed) learning difficulties, I can remember getting into trouble very frequently as a child for choosing behaviour which did not feel inherently bad, but which simply did not conform to what felt at the time like a never-ending and obsessively long list of school rules.

And so it is that slowly but surely creativity and curiosity too often begins to take a back seat as we learn to conform to the expectations of others. This of course is to the great detriment of learning at Greater Depth. This happens on three levels:

Firstly, on an individual person-to-person level. This may be a parent, a teacher, a friend or perhaps someone with whom we are interacting only briefly. In any case, we quickly learn to stifle what might come naturally as we learn to make that interaction as smooth as possible. Learning that "*Miss is a really strict teacher, she shouts if you ask questions*",etc. inevitably stifles my curiosity about the subject of her classes, and my main objective becomes survival rather than learning.

Secondly, our tribe or peer group has a huge influence on us as we grow up. I might be keen to try a particular style of shoe but knowing that my peers might mock me if it doesn't fit an 'acceptable current style'

I may think twice, and restrict my choice to styles which I hope will not lead to me being ostracised.

Thirdly, we begin to learn that the wider world has certain expectations. The type of car you drive, the speed at which you drive it, the cost or style of clothing you choose, the television you watch, the social media posts you make, your sexuality, your choice of friends, the music you listen to, the political views you express and more. Other people are very quick to tell you that you are wrong, both directly and indirectly. By which they can only mean you think differently to them, of course.

Of course, all of this is entirely natural throughout the whole of humanity, and sometimes helps us avoid mistakes, (speed limits etc.) but many societal norms, specifically those which are more about judgment than well-being, pose a significant threat to our creativity and curiosity.

The difficulty for children growing up is that it is very hard to distinguish between rules like these:

- ‘don’t go too close to that open fire’
- ’don’t draw on that piece of card’
- ‘don’t wear socks with sandals’

The first is an excellent rule; it is aimed at keeping us from getting burnt. The second might well make sense situationally, as you may be planning to use that piece of card for something later. As for the third: what is the worst that could happen to anyone who makes what we regard as a fashion faux-pas?

So because it is tricky to distinguish, we end up assuming that rules are, well, rules. We either conform to them all or we style ourselves as ‘rebels’, and happily break them.

A Possible Solution

So if an over-imposition of rules is as big a threat to curiosity as I am suggesting, how do we avoid the pitfalls while still maintaining order? We cannot simply banish rules and allow children to do whatever they wish. This is progressive nonsense and although espoused by a few, it has been repeatedly demonstrated that it is not a sensible course to take. Children need guiding unto they are sufficient clued up about the world to decide for themselves how to behave. See Lord of the Flies, by William Golding for a stark reminder of this!

The answer lies in deliberate planning. Deliberate planning sounds like a tautology, I realise. After all, how can planning be anything but deliberate? It depends. I used to plan lessons, then I moved towards planning tasks and now I prefer to plan learning itself. This is not just semantics, but a really important distinction. The reason for this is simply that I have a range of activities that I can draw on, either from experience or various printed or online sources. But one key to planing learning is to realise that it is a messy old business. While we can predict an approximate path, that is as much as we can or should ever attempt. Learning is almost never linear.

"Learning is a messy old business. While we can predict an approximate path, that is as much as we can or should ever attempt. Learning is almost never linear.’

The idea that you can plan a whole term of work simply does not stand up to scrutiny, Instead, what if we first thought about the core big ideas of a concept, and concentrated instead on why they are important, how they interconnect, what skills children will need to learn, what techniques they will need to master, and finally what tasks and questions we might use to engage children and give them the practice they need to develop their understanding? How powerful might that be? Headteachers, if you are still requiring teachers to 'hand in their maths planning' (whatever that even means) please stop!

OFSTED recently announced that from 2019 their focus will be on curriculum, and this has inevitably led to schools re-examining, and in many cases re-designing their curricula. While this may in some cases be out of fear of criticism, it nevertheless affords us a considerable opportunity to think about what learning with 'Greater Depth' might look like.

My advice when thinking about deepening children's mathematical experience, would be to consider these five fundamental questions:

- Do my children see maths as a key and integrated part of their lives, and not just something they do after break?
- Do my children get a chance to 'do' mathematics, or do they just repeat somebody else's question?
- Do my children see maths as a creative and social endeavour or thing more than rules to be memorised and followed?
- Do my children understand maths as a language with its own words and pictures, and do they practice communicating in it?
- Do my children know that it is OK to wonder?

Currently in the United Kingdom we place such emphasis on high-stakes testing that it is little wonder children get the message that ticks are what we need above everything. This needs to change if we are to see our young mathematicians truly think deeply about their mathematics.

2. Metacognition

“I am aware of, and able to explain what I am thinking about."

The word ‘metacognition’ has gained in popularity in recent years, yet it is far from new. In essence, metacognition refers to the conscious act of being aware of and understanding one’s own thinking; in other words, it involves being able to think and talk about the thoughts you are having.

Encouragingly, metacognition skills *can* be specifically taught. It requires patience and practice (though what doesn’t?) and there is evidence that people who are able to regulate and adapt their thoughts can perform more effectively in both familiar and unfamiliar situations. Cognitive Behaviour Therapy, a common mental health intervention, has been proven to be extremely effective, and it is 100% based on metacognitive techniques.

A review of current literature suggests that metacognition is a very simple thing dressed up as a very complex one. For us, it boils down to our young mathematicians being aware of the thoughts they are having and appreciating that they are a part of the reasoning process. “Think about what you are thinking about” is my way to explain it, or to paraphrase John Mason: “Are you aware of where your awareness is at?”

Metacognitive strategies are the mental behaviours we employ to monitor our thought process, and to amend and control it where necessary, for example if we think an answer is incorrect. There is evidence that children as young as 3 are able to behave in this way.

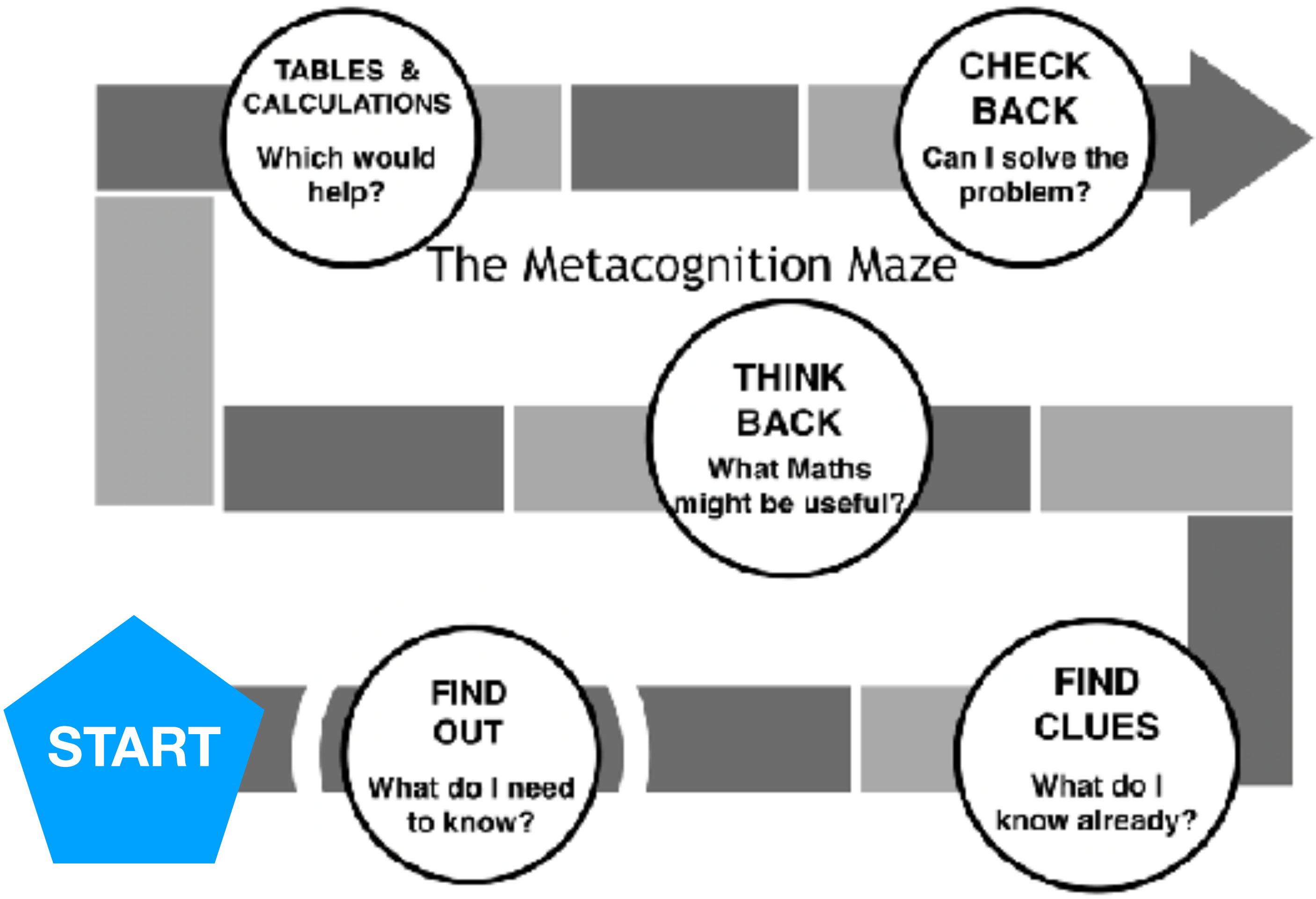
The word itself has entered the lexicon of common educational parlance relatively recently, yet the underlying ideas are much older. Ideas such as like contemplation, comfort zone, challenge zone, reflection, self-appraisal, debate, self-efficacy, explanation, and yes, awareness have been part of our thinking for centuries. Perhaps then, metacognition might be a useful umbrella term that provides a framework for thinking about how these might usefully manifest in our classrooms.

A 2019 report from the Education Endowment Foundation, entitled 'Metacognition and Self-Regulated Learning' lists seven recommendations for ways in which teachers might support students. These are:

- Acquiring professional understanding and skills
- Explicitly teaching metacognitive strategies
- Modelling our own thinking
- Setting appropriate levels of challenge
- Promoting and developing metacognitive talk
- Explicitly teaching how to organise and manage learning
- Schools supporting teachers to develop and apply the above.

This is a good list but needs exploring in, if you'll excuse the pun, greater depth. For example, what ARE the professional skills required? What DO metacognitive strategies look like? What IS metacognitive talk? For some more detail from the report, please see the bibliography where there is a link to where the document can be freely downloaded.

A word of caution. Guided practice, in which correct methods are modelled and then students try with gradually reduced levels of support is a proven methodology. It is highly favoured in successful Asian jurisdictions such as Singapore as well as more traditional UK classrooms. *Also* though is the approach that leads children to think, make conjectures and spot patterns. It is an entirely false dichotomy to say that either is the only way. Some people are very wedded to one, to the exclusion of the other - I think this is sad, as it is the children who will miss out by not experiencing both approaches. In an attempt to make the ideas of teaching metacognition in a classroom more tangible, I have produced the following diagram.



As you can see, it is a simple five step process (the EEF report has seven steps) which I believe has genuine potential, if used correctly, to help students to think about what 'thinking' and 'doing' they need to do, I have been guilty myself of simply asking my students to 'think', without stopping to explain what I mean by that term. In essence, *thinking is a conversation with oneself*.

A final thought for this section: people who talk to themselves are sometimes considered strange, but this is nonsense. EVERYONE talks to themselves - most of the time, that conversation is internal, but if that thought process is voiced aloud, in what way is that strange? In fact, it might even be the case that thinking out loud is a superior form of personalised dialogue, as the brain has an extra mode of input available, namely the ear. For a really, really in-depth look at thinking being personalised dialogue, I recommend 'Thinking as Communicating' by Professor Anna Sfard (2012). This is not an easy read but it is a thorough treatise based on a life's work and I can safely say that it changed my thinking about thinking very significantly.

3. Making Connections

The introduction to the National Curriculum (2014) contains this hugely important paragraph:

> *"Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas. The programmes of study are, by necessity, organised into apparently distinct domains, but pupils should make rich connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems. They should also apply their mathematical knowledge to science and other subjects."*

There is no mention of what these representations might be, but fortunately the work of Zoltan Dienes and Jerome Bruner provide plenty of clues.

Consider a child who has two blocks, and puts a third block alongside them. By counting all of the blocks in the group she may ascertain that she now has three blocks. This is what Bruner refers to as the '*enactive*' stage of maths; literally representing mathematics as something that can be done physically 'done', in this case addition. At the age of 14 she may be adding algebraic expressions but crucially, the mathematical idea is identical to the one she was doing aged 4.

In current parlance teachers often refer to the doing as the '*concrete*' phase, and while this is broadly true as a working definition, it is not quite the entire picture. If we break this process down to a more granular level, there is a natural progression. At first, the child will simply aggregate. That is, she can count two and count the other one as a separate group, but when combining them, in order to discover the 'total' (a difficult concept) she will now count the whole pile, starting at the beginning, as 'One, two, three."

Only several months later, around the age of 5, might she begin to count by calling the former pile 'two' and counting on 'one more' to three. This is known as the process of augmentation, and is a vastly more efficient method of addition than aggregation. Consider for example, how long it would take to add 8 and 2 using each approach.

These early examples of addition are vital and cannot be rushed. One reason for this is that around 6 or 7 years later, that same child will meet expressions such as:

$$2x^2 + x^2y$$

If they spent sufficient time time when young realising that two of something plus one more of the same thing gives a total of 3 of that thing, then realising later on in Key Stage 3 that

$$2x^2y + x^2y = 3x^2y$$

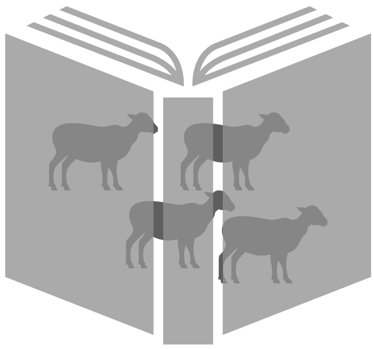
will be far more intuitive for them, as they are now able to replace objects with an algebraic expression.

One successful way to support children make the transition from aggregation to augmentation is by use of something familiar such as farmyard animals in barns. More practical of course would be small toy animals and shoe boxes or even just open books, which in the author's experience make excellent barns.

Here is how this might play out in an Early Years setting. We place some (let's say 4) sheep in a barn, and ask a child how many there are. If the child can count correctly - don't assume, check - we then put the roof/lid on the barn and ask how many there are now. If the child is not sure - and this is entirely possible at the early stages of counting, allow them to remove the lid and count again.

Replace the lid and repeat this wondering. This may need to happen a few times before the child becomes satisfied that *the number of animals in the barn does not change when the lid is put back*.

Here Is a barn (yes it is!). In the barn are four sheep and even though they cannot be seen, children can peek behind and convince themselves that the barn contains exactly four sheep.

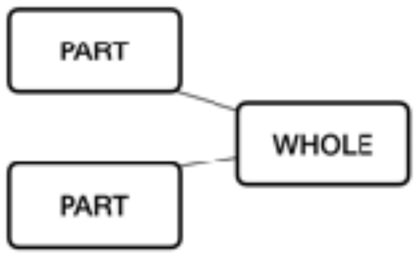


An effective step here is to write '4' onto a sticky note and stick it on the front of the book as shown:

This is powerful as it allows children to make the connection between the written numeral '4', the word 'four', and the hidden quantity. In addition, this process is supporting their developing understanding that '4' represents the total number of objects, and not merely the final one in the group to be counted.

The next step is an exciting one (for children and adults!). Give the child another toy sheep. Ask them to place it in the barn and talk about how many sheep are now in the barn. Because we have made a big play of putting sheep in the barn, they may still want to count, and that is fine. However watch out and listen for children who are able to use sentences like ‘one more than four is five’. You can even add more sheep or take sheep away once they understand the rules of the game. Better still ask them to play against each other by saying how many sheep they are putting in or removing and getting their partner to guess how many sheep are in the barn subsequently.

Of course this is just one example of how children make connections. Another connection in mathematics that can and should be made while children are still around the age of 5 or 6 is the relationship between addition and subtraction. The UK has made increasingly good use of the part-whole model in recent years and this is an excellent example of how we can encourage greater depth thinking.



This image, reproduced here from ‘Bar Modelling in Key Stage One’ (2018) and also widely used elsewhere is well known. For best results, I have found that laminated A3 versions of this work best as children can use both written figures and practical resources to count with.

Why is this so important? From an early age we are told there are ‘four operations’. This is not particularly helpful if we wish children to make connections. There are many ways to combine numbers, but addition, subtraction, multiplication and division are the building blocks of all others, hence the understanding that has grown up around them. Of course as children get older they see that numbers can relate to each other in other ways such as the use of index or logarithmic relationships too, but as this book is only conferenced with mathematics at primary level we will focus only on the first four.

Unfortunately they are taught as separate ideas. But what if, instead of seeing them as four operations, we saw them as two relationships? This is a good example of what Greater Depth can and should look like.

It involves far more than merely pointing out that they are inverse operations; it is understanding that when in pairs, they are symbiotically linked; one cannot exist without the other.

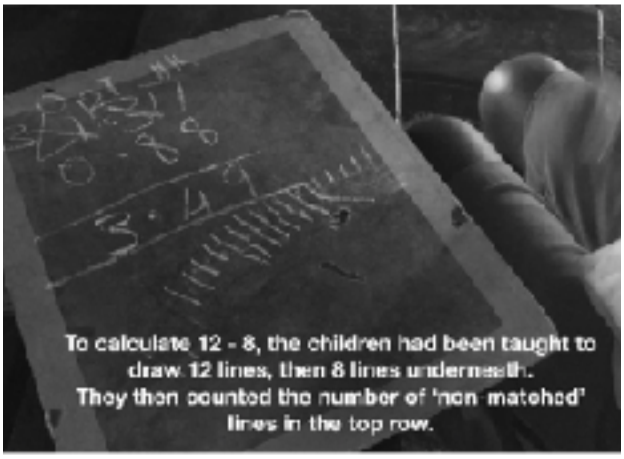
This is a crucial idea in the development of both written and non-written calculations, but particularly perhaps non-written.

Think about the following questions:

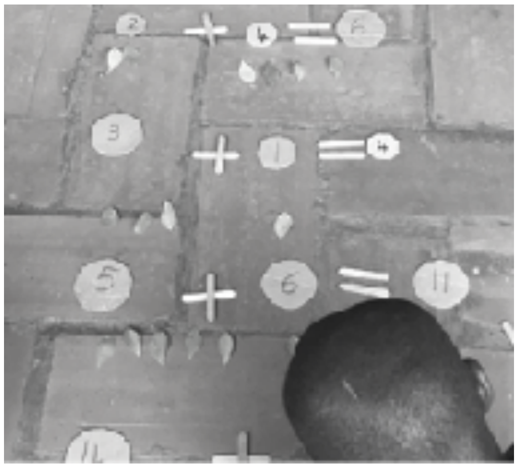
$$24 \div 12 = \square$$

$$8 - 7 = \square$$

In the UK we would hope that many children will use the fact that 12 x 2 = 24 to solve the first, and the fact that 7 + 1 = 8 to solve the second. But what if those relationships are not well understood? I was recently watching a lesson in a Ugandan school, where children had never been taught to make connections between addition and subtraction. Here is how they correctly but inefficiently found the size of 12 - 8. I asked them to find the value of 10-9 and they opted for the same method. These children were 9 years old. Clearly we are



doing children a disservice if we do not allow them to make connections, and here is where number lines and part-whole models would play their part. In the school’s defence, they had almost no resources, but they still had bottle tops and small stones with which they could have made this a more concrete experience for the children.



Compare the image above with this one, taken in a model nursery school where teachers have been trained in the importance of a CPA (Concrete, Pictorial, Abstract) approach.

How might the part-part whole model help, and what other resources might we suggest in this case to make 10-9 a very much more straightforward question to answer?

Firstly, to really begin to understand the addition and subtraction relationship, we need to start right back at simple counting. Although counting is far from simple, of course. Initially counting is nothing more than memorisation, based on thoughtful imitation and repetition. Later it becomes about one-to-one correspondence, and later still this becomes about more than mere nominals - a number takes on a new and richer meaning specifically with respect to the size of a set.

For example, think about a child who successfully counts three objects by saying ‘One, Two, Three” as they touch each separate item. In their basic level of understanding, it is as if they are saying “cow, sheep, horse”, the third item being the horse. But the idea of ‘counting all’ says that the WHOLE group is in some way ‘horse’. This is the important idea of cardinality, and imagine how much of a change that is for a young child!

So TIME is very much key here. If we do not allow children in the Early Years enough time to explore ordinals (first, second, third etc) once cardinals (one, two, three etc) are secure, it will be much harder for them to move from counting to calculating in Key Stage 1. Yet I am concerned that EYFS experts are not always listened to at the highest levels and the Early Years is becoming too formalised too soon. Learning takes time. Being able to replicate a procedure from memory DOES NOT equal learning!

To conclude section 1, let us move on to briefly consider the power of open questions. Many of these will be expanded upon in Section 2.

4. Open Questions

Research from the late Ted Wragg at Exeter University found that in primary schools, about 57% of questions that children were asked in primary school were of an administrative nature:

"Have you hung your coat up?"
"Are you having a school dinner?"

A further 35% were questions requiring factual recall

"What are six sevens?"
"How many wives did Henry the Eighth have?", etc.

This left only 8% of questions as 'open' questions, which we can define here as questions which require more than simply accessing a memory.

This is a worrying statistic, and while hopefully this figure has risen slightly since Wragg's research was undertaken, the chances are high that we could benefit from making a determined effort to ask more open questions.

Fortunately, this is a relatively easy habit to pick up. Instead of asking, for example, "*What are six sevens?*", it is easy enough to ask "*If the answer is 42, what times tables might the question have been?*", perhaps followed by "*Is that the only possibility?*"

Of course, this tests the same factual knowledge, but it also invites the student to consider pattern and structure. For example, if I recall the possible answer of 6x7, can I double one number and halve the other to give 3x14; and will that always work?

But what other examples of open questions might be effective in the primary classroom (or indeed any classroom)? There are any number of great questions for promoting deeper thought, but before we go through some it is important to note that these kind of questions are certainly *not* at the expense of factual recall - factual recall is essential! Children require a bedrock of knowledge and skills *before* they are able to think in deeper ways about mathematical ideas, and this will always require an investment of time in rehearsal of recall of facts and skills. If that sounds somewhat old-fashioned, then so be it, but experience overwhelmingly suggests that children think differently (and specifically more deeply) about something in which they already have a basic grasp of facts and processes. This may be because fluency means they are not using up valuable working memory on the basics.

With that important caveat, here are some effective open questions to try. Don't feel constrained by the wording; it is the spirit of the question that is salient here.

Is that the only possible answer? How could we be sure?
What might we change to make this question easier or harder?
Does this remind you of anything else we have done?
How would you convince someone who had a different answer?
Will this always work?
Have you found all the possibilities?
What would happen if…?
Can you prove that…?

If we know…. what else do we know?
How could we find out...?
What numbers do you think it might not work for?
What is the pattern?
Can you explain why this works?
Can you convince me/your partner that...?
What could we think of that would disprove this?
What is the same, and what is different about…?
Is this statement always, sometimes or never true? What else would we need to find out?
What else can we deduce if this is true? Why is this wrong?
What might I have done to ....to get…?
What else might we ask or wonder?
Is there an alternative answer?
Is there a faster method?
How many other ways can you find to…?
How could we best organise this information…?
Why doesn't this always work?
When does it work?
What do you predict will happen if…?
How would you explain to an alien that...?

This sort of question needs planning carefully. Do you want children to think on their own, in pairs, small groups or as a whole class? Whichever you choose, and each have their merits. it is important to allow some time for individual reflection initially. This is so that each member of whatever group size you decide upon has a greater chance of having some thoughts to 'bring to the table.' We probably all experienced lessons as children in which the teacher asked a question, a quick-thinking pupil got the answer, and the teacher moved on. We are unlikely to have gained much, if anything, from this exchange, though the teacher may at least have felt encouraged that their question was answered correctly.

As you work through the remainder of this book, think carefully about how you can use open questions to encourage children to pause, reflect, discuss and share their ideas. Section 2 will unpick many of the ideas we have discussed so far, using ten simple examples. They are not designed to be sequential; it is perhaps best to try out one of two that you feel would work well with your own group of learners, and reflect on what happened.

Enjoy!

Part 2

Strategies for Encouraging Deep Thinking

In this section you will read about ten different strategies. You may very well be using some of them already, so feel free to skim those. It is almost certain though that others will be new to you; reflect on them and try them out.

For each strategy, you will see the ‘in a nutshell’ section, a detailed breakdown, and then two possible examples, one from KS1 and one from KS2. This is intended to help you to find practical applications of the strategies in your own classroom.

We start however by considering a typical classroom scenario. Whenever a teacher asks a question, there will be a range of responses. The ten strategies we are outlining are designed to ensure that ALL children are able to engage with the mathematics, and develop their own thinking without the need to hide or behave in a defensive way.

Picture the following fictitious scenario which you may well recognise from your own classroom. The teacher asks a question requiring a 'correct' answer. This might include such things as '*What is 2+3?*", "*How many sides does a pentagon have?*", "*What are the factors of 40?*" and so on.

Looking around the classroom we can see five groups of children. Some whom we will call **group one**, are looking completely blank. They have no idea what the answer is, nor do they have any intention of trying to figure it out. They have learnt that by staying very still and showing no sign of engagement, the teacher will choose someone they feel has more chance of moving the lesson on with the required correct answer, and thus they will successfully have avoided awkward conversations and looking stupid.

There is a **second group** of children. They are also quiet, but their faces suggest that they are at least thinking about the problem. I say 'suggest', because in fact they have perfected their 'thinking face'; a narrowing of the eyes, a slight tilt of the head, a wrinkling of lips or nose, and a slight upwards glance as if attempting to recall some vital information.

Next we find a **third group**. These children are hard to spot, because they can often look very similar to the first or second groups, yet their internal thought process is very different. They are genuinely processing the question, slowly and thoughtfully, and after some time, possibly twenty seconds or so, they will be ready to offer a well-considered answer. We will return to these children later.

Group four are often the teacher's favourite, though clearly this must never be publicly acknowledged. Having thought about the question for a micro-second, their hands now shoot into the air as if their lives depended on it. Their enthusiasm is encouraging for the teacher, and brings great energy to the lesson. Any observer would note how keenly these children approach the lesson, and credit the teacher accordingly. The teacher will be tempted to select one of this group to answer the question. Some members of this group have graduated from group three, and it is interesting that their hands may be raised somewhat less effusively than the other members of the group.

And finally, we meet **group five**. Not for them the irritating wait for possible selection. The key thing is to demonstrate their superiority and gain the respect of their peers and the teacher. Not for them the raising of the hand and praying to be the chosen one. Their preferred strategy is to call out the answer as soon as they have it, and being fast is more important than being correct. Being the first to answer is of course vital, as it feeds their self-esteem and shows that they are indeed the top dog and thus worthy of recognition.

If you are new to teaching you may not have met all of these groups of children. Rest assured however that they will all become very familiar to you in time. So let's now consider who is doing the learning in this scenario.

Group 1: Clearly these children are not learning any mathematics. They feel, rightly or wrongly, entirely unable to engage with the question. There are a few possible reasons for this.

Possibly their confidence has taken such a battering due to past struggles with mathematics that they simply cannot risk another failure, so they feel safer not attempting it. It is easy for us to judge, but with a moment's thought we realise that adults behave like this too. Is there a road you avoid? A colleague? A chore you never get round to? These children need their confidence working on *before* they will be able to learn any mathematics. As well as huge amounts of skill and patience, this requires a significant investment of time, and given the pressures on schools, often this is simply not available in sufficient measure. Or perhaps they have missed some vital building blocks - mathematical understanding builds in a spiral way. So much of what we learn is only possible if we have mastered certain key concepts previously. If this is the case, no amount of encouragement will have any effect; going back to basics is required.

Going 'back to basics' is often used in a vague and meaningless way, but in this case I define it to mean the pre-requisite skills and ideas necessary to progress to the current level. For this reason I am very much in favour of grouping children according to their level of conceptual understanding, rather than the less dependable system of whether they were born on 31st August or 1st September. It may be less labour-intensive for schools to agree that they will spend a week on place value during which every child will work at their own level (meaning all classes need to take maths at the same time, of course) and mixing up the classes across year groups. Bold? Perhaps? Impossible? Far from it! Alternatively, this could happen one day a week on an agreed topic, where progression through that topic has been clearly identified. This is actually easier, not harder, under the current (2014) curriculum, as objectives are now end of key stage rather than end of year, affording us more flexibility.

Group 2: These children are not learning any mathematics either, and in some ways are in a worse situation than those in the first group. Those in group 1 are at least making no pretence that they are wrestling with the question, making them easier to target, and offer scaffolding or intervention as necessary. Group 2 children however do not like to admit that they have no idea how to proceed, and are very unlikely to ask for help. It is vital that we challenge these children by asking what they are thinking about a problem. It may well be nothing, but at least we are able to assess this by the way they answer. Much has been spoken recently about the 'Growth Mindset', much of it very misleading, The instigator of the original research, Dr. Carol Dweck, in no way implied that everyone could be brilliant. She simply demonstrated that if children are praised for their *effort* as opposed to their *ability* they are more likely to take risks with harder material and ultimately outperform those children who have been too anxious to push themselves for fear of losing their 'clever' label. We have for some time now, due in no small part to the high-pressure testing regime, become fixated with getting high scores - worshiping at the 'temple of the tick'. Group 2 children need a confidence boost, and cultivating an 'I don't know, let's try to find out' attitude and culture is likely to have a positive impact on these children over time.

Perhaps more than any other group, group 2 are likely to benefit from the strategies in this section of the book. One final caveat about group 2: if we wrongly ascribe a child who genuinely IS thinking about the problem (i.e. a member of group 3) to this group, then no matter - the class will get to hear, and potentially benefit from, their thoughts, should we call on them to share.

Group 3: Paradoxically, these children can be the hardest to manage, yet are often behaving in a very mathematical way. Why hardest? Because our natural human temptation is to fill silence. I have struggled with this myself as a teacher. I want to move things on, to get answers and share thinking, yet I have not left enough time for any thinking to occur!

Thinking time is valuable and all children should be given plenty of it, before they are asked to share their ideas. So how much is plenty? That depends on the complexity of the question, of course, but in general we should remain completely silent for at least ten seconds after asking a question, and very often longer. It is hard for children's brains to
think carefully about a problem and at the same time listen to the teacher attempting to '*clarify*'. This is an unresolvable conflict for the brain as it tries to listen to both internal and external inputs and can inevitably do neither.

You can demonstrate this easily - try asking a friend a maths question of reasonably complexity and then talking to them as they try to work it out. It will most likely prove immensely difficult either to solve the maths problem or recall the facts.

So group 3 children are in fact great role models. All they need from us is silence, time, and very possibly a pencil and paper. They will thrive in an environment where speed is not praised over efficiency, but reflection and explanation are valued highly and openly.

Group 4: I love these children, though perhaps in the same way that people love adorable and tail-wagging puppies. Specifically, in a

'*Ten-out-of-ten-for-enthusiasm-but-let's-all-calm-down-and-think*' kind of way. They are in some ways the easiest group for teachers to deal with, as they are already a) motivated, b) thinking about the question, and c) confident enough to have a go.

The trick with group 4, as with actual puppies, is to have firm rules in place, and train them over time to pause (paws?) to reflect. The use of talk partners is an excellent solution for these children, as it provides an outlet for their enthusiasm, and also allows the communication of knowledge by more than just one person. 'THINK-PAIR-SHARE' is an ideal way to organise this. One particularly good strategy for these children, that also works well with groups 3 and 5, is to insist that *any answer given must contain the word 'because'.* If you have never tried this, you will be amazed at the improvement in children's mathematical vocabulary in a matter of just a few weeks as they strive to gain recognition from a partner. Try this and listen to the difference! I am indebted to the wonderful Hamsa Venkat, whom it was my pleasure to meet in Johannesburg, for this powerful tip.

Group 5: I have to confess I have a soft spot for group 5 children, partly because although my memory of my school days is not good, I suspect I was something of a group 5 child so have some sympathy. Some of the most incredibly gifted young mathematicians I have ever met belong to this irritating but inspiring group of children.

It was my great privilege many years ago to teach Daniel, when he was in Year 5. Daniel was one of those children who loved everything about school; music, art, academics and sport - and was NEVER able to curb his enthusiasm. He was a delightful child, but quite incapable of remaining silent in class for more than a minute at a time, such was his passion for maths. This was great to see, but asking questions to the class was almost impossible, as Daniel seemingly could not (and indeed did not) wait to call out the answer to orally posed questions. There were other bright children in the class but they struggled to get a word in edgeways. I realised I needed to do something to involve everyone without curbing Daniel's natural enthusiasm. My mother used to say, "engage brain before opening mouth", but this wise advice did not apply; Daniel HAD applied his brain before opening his mouth - it just took far less time that it took the rest of us.

It is undoubtedly this group of children of whom I was thinking when I wrote some of the strategies that follow. The ninth strategy, 'Who Heard", in particular, worked a treat for Daniel, and the rest of the class benefited.

I hope you enjoy them (you may well already use some of them) and that they allow EVERY child, whatever their current learning looks like, to benefit from thinking about mathematics in greater depth. They can be read in any order and will work both independently of each other or in combinations - don't be afraid to experiment. Go and enjoy your maths lessons!

1. Why is this the answer?

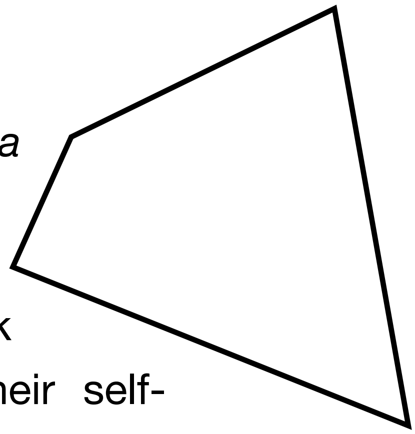
In a Nutshell: Pose a question, give the correct answer, and then ask children to explain *why* it's correct, rather than requiring them to spend time working out the answer.

There is much to commend this technique. First, it sends a strong message that yes, answers are important, but that they are not the be-all and end-all.

Second, if we return to our 'five groups' scenario, many children tend to feel they have permission to stop thinking the moment another child calls out the 'correct' answer. It is all too easy (*I know, I've done it far too often as a teacher*) to actually shut down thinking by acknowledging a correct answer too quickly and moving on. By asking a very different question, specifically "WHY is this the answer?" we take pressure off those children in groups one and two who wouldn't know where to start with "WHAT is the answer?" Too often a lesson segment can be nothing more than the teacher asking a question, a child providing the correct answer, the teacher asking another question, another child providing the correct answer, and so on. This is of no use whatsoever to groups 1 and 2, and the 'why' technique affords them an opportunity to access the learning, especially as the explanation is coming from their peers.

Perhaps most importantly, it emphasises the crucial importance of *understanding* within mathematics. This aspect of learning will be common to many of these ten strategies. Nothing feels quite as demotivating as doing something you don't understand just to keep the teacher happy. This is just mindless ‘performance’ and not to be confused with procedural fluency which is a good and desirable thing.

Key Stage 1 Example



“I could say this was a quadrilateral but not a square. Can anyone think WHY?”

Now, the whole class have something to work on. What’s more, nobody needs to risk their self-esteem by possibly naming the shape incorrectly (we now know from the teacher what it is and isn’t after all), so that pressure is off. But there is also an invitation to think about WHY. And now that we know it is NOT a square, we can look for features of a square that are not present - this might be the right angles, or equal side lengths, depending on our subject knowledge. But again, ALL children are now capable of engaging with this question.

Possible answers might be:

- “*It doesn’t have the right size corners*”, which of course would need teasing out a little, and correct vocabulary modelled.
- “*The sides are different lengths*”
- “*The angles aren’t the same*”
- “*There are no right angles*”. In fact, there is one, though this needs pointing out carefully.

Key Stage 2 Example

"*Ravi reckons the only even prime number is 2. Can you explain why he's right?*"

Again, we are given the facts. We can now consider all the other prime numbers (odd) and think about their factors. Or we have forgotten, in which case we can work with a partner. Either way, we have something to do, and the removal of pressure. Of course we are looking for something along the lines of "*All other even numbers are multiples of 2. This means that they have at least three factors so cannot be prime.*" Perfect.

2. And another

In a Nutshell: Ask children to give an example of something, such as a pentagon a multiple of 11, or a pair of numbers that add up to 8. Make a note and then ask for another example. Keep going until children find as many examples as possible.

This is a simple but useful technique for helping children think deeply about mathematics rather than regurgitate. For example, if you ask them to recall a fact or show an example, there is little to prevent then accessing their memory for an example that would fit. But by asking for another, and another, you are requiring them to think not so much about their example, itself, but about the features that ensure it fits the given criteria.

Key Stage 1 Example

"Write down a calculation with an answer of 10." In the initial stages as many as three-quarters of children in my experience may opt for 5+5. And this is fine, but it is using only memory. By asking "*and another"* children are forced to move away from memory and instead focus on the idea of 10 itself. Most, if not all, will then go to another whole-number addition. perhaps 9+1, and that is fine. Continue to ask for 'another'. The skill here is knowing just how long to wait until you ask *"Does it have to be addition?"* (Hint: probably longer than you think). If children have been given enough time to think and come up with examples, then it is highly likely that subtraction and multiplication will start to appear. Perhaps even division may turn up; whatever the children do, record it on the board. In this way the experience is a shared one, and you can always reference this rich seam of greater depth thinking once the

task is complete. Ih this way children are learning from each other - the whole really does become greater than the sum of its parts here.

A key thing to notice here is how effectively differentiation is at work. There is no '*genius table*' or '*muppet table*', grouped by prior attainment, often worryingly referred to as '*low ability*'*. Everyone is working on a simple question, yet all are given opportunities to think about it in different and potentially ever more complex ways - that is the true meaning of greater depth.

Suppose a child comes up with 100-90? Or a half add 9 and a half? Just imagine the discussions that would ensue!

Key Stage 2 Example

"Write down two numbers whose product is 20."

Once again, there is plenty of differentiation and opportunity for deep thinking. Inevitably 4 x 5, 2 x 10 and 1 x 20 will be offered first. Many children will not initially realise that there are other categories different 'groups' of answers; two negatives, a fraction and a number larger than 20 and so on. By requiring them to continue to find new unique examples, they will be forced into exploring their understanding of multiplication. *Technically primary children are not required to multiply negatives, merely add and subtract them, so it is unlikely but not impossible that they will choose an example of this kind, but ⅓ x 60 is entirely within their reach however.*

Asking children to draw a rectangle with an area of 20 (and another) would be a good alternative question - the numbers are the same but there is nice variation in the concept: this is greater depth.

** If you are using this terminology, please have a think about its implications. I prefer the term 'slower graspers' as ability is an impossible thing to pin down and quantify. Some children take longer to get ideas, but that does not necessarily make them less able.*

3. Same and Different?

In a Nutshell: Give children two or three examples of numbers, or shapes, or calculations. Ask them to find as many similarities as possible between the examples, and then to find as many things which distinguish each example as being unique as they can.

It is probably not overstating the case to claim that learning to compare and contrast according to mathematical criteria is one of the fundamental tools of mathematics. Focusing on features that vary or remain constant is invaluable. Yet too often we do not allow time for simple 'noticing'. We rush straight into problems without stepping back and getting a big picture view, which might help us to solve problems faster. One type of question that they are fond of asking children in Shanghai, for example, is

(2/3 + 1/5) + (4/5 + 3/7) + (4/7 +1/3) = ?

Jumping in with our 'make the denominators the same' sledgehammer will create a huge amount of work, but removing the brackets and pairing similar fractions yields a far simpler solution. Thus, by building the habit of noticing, we are doing children a huge favour in terms of their future efficiency. Asking what is the same and what is different about the fractions in the question above would save children potentially several unnecessary lines of working.

Key Stage 1 Example

What is the same about all of these: 1   2   3   4   ?

Do not expect great insight in the initial stages, though you may be surprised. Typical answers are '*they are numbers*', '*they are small*', '*they are less than 5*', and so on. You may well need to scaffold in the early stages. I also find it is easier to ask this question in two parts. For example, you might ask "What is the same about all of these?" And when that had run it's course, only then would you say something like "*How could you finish this sentence: 2 is the only number in the list which…*"

Key Stage 2 Example

What is the same about all of these words?

TETRAHEDRON QUADRILATERAL CUBE SPHERE

The range of answers children provide to any 'same/different' question will give you a lot of information about their prior knowledge, so this is an excellent task to undertake at the beginning of a new topic. But it works equally well as a summative assessment. Comparing children's answers at the beginning and end will give you a sense of progress of course.

Apart from being shape names, they have little in common, so more interesting will be the second section, in which we encourage the noticing of difference. What is unique about…a tetrahedron? A Quadrilateral? etc. By giving children words rather than the actual shapes (we are forcing them to think about meaning rather than just observe, and this engages the brain in a deeper way. Finally, in either key stage you could also show images, and play 'odd one out' Here is an interesting sample:

4. Always, Sometimes Never

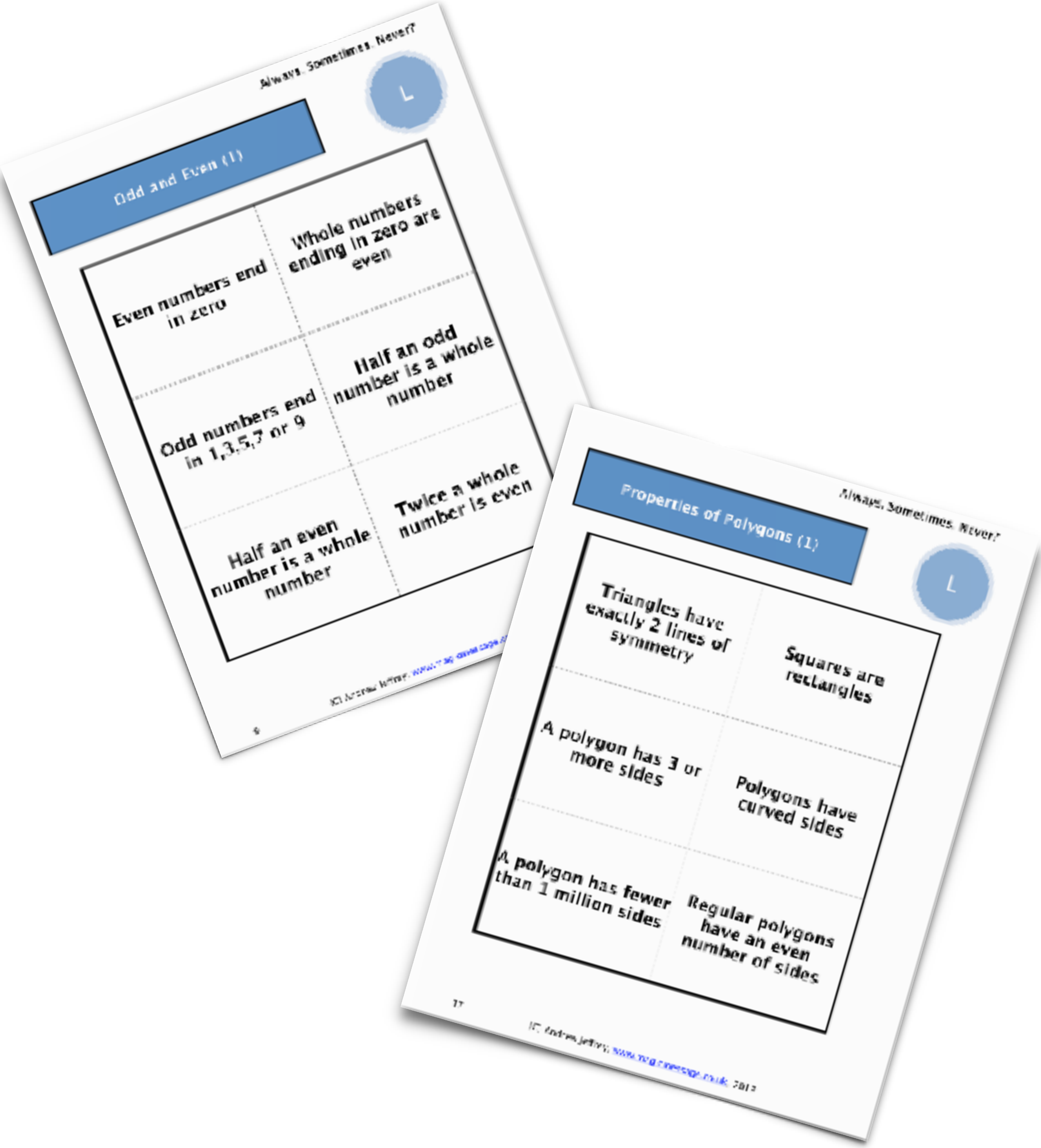
In a Nutshell: All statements are either always, sometimes or never true, so give children a series of statements and ask them to decide which of the three words applies - is each statement *always*, *sometimes* or *never* true?

Over the years I have realised the power of this task lies in the classroom management. Every group of children in the scenario that opened this section will be given opportunities to develop and deepen their mathematical understanding. In a similar way to strategy 3, this can be used formatively or as a summative assessment, but its real benefit is getting children to communicate their thinking to each other.

In both key stages, the Always Sometimes Never (ASN) task can be made more effective by using the tips that follow.

Firstly, prepare a set of statements on cards, one statement per card. (See Key Stage 1 and 2 examples overleaf, taken from my 'Always, Sometimes Never' e-book, available from andrewjeffrey.co.uk, or develop your own based on the topic being studied.

Consider the flexibility of this technique. There is no area of mathematics where you cannot find suitable statements to try. It is particularly useful for thinking about what are referred to as 'boundary cases' or exceptions to rules'. Also, the answer can also depend on children's level of knowledge - if they have not yet learnt about negative numbers, for example, a statement such as 'Adding makes things bigger' will feel like 'always' though of course adding (for example) negative seven will not do so.



To maximise the effectiveness of this strategy, the way in which we set up the task is of paramount importance. I have discovered seven different ways to implement ASN - you may find another. Here they are:

- a) Whole class (1): Print out large versions of the cards to hold up, or display them on an IWB. For each statement, discuss it as a class, then decide in which category the statement belongs.
- b) Whole Class (2): Display the cards as in (1), but this time children work in pairs or small groups to decide on their answer. Each pair or group should then vote by holding up a pre-printed card or whiteboard showing a large A, S or N card.
- c) Whole Class (3): The statements are printed out onto A4 cards and each group is given a card. Three areas of the classroom are allocated to represent 'Always' Sometimes' and 'Never' and children should decide where to stand. To avoid congestion, only one person from each group needs to take the card to their chosen location. More than one group of children can have the same statement, a leading to some high-quality debates. It is sometimes interesting to give ALL the children the same statement!
- d) Pairs : Give each pair of children a set of cards. They must discuss the statements and sort the cards into three piles. Once each pair has finished, they can discuss their answers with another pair sitting nearby. They can change their answers but only if they can give a convincing reason why they have done so.
- e) Groups**:** Distribute a set of cards to a group of children – they must read out their card to the rest of their group, and the group must then decide in which of the three categories it belongs. It can only be placed once EVERYONE agrees.

- f) As a summary to assess learning. For example, if you have been working on percentages, you might want to generate a set or statements that refer specifically to that topic and ask children to sort and discuss the statements.
- g) Independently: Cut out the statements and stick their solution card (on the facing page) to the back. The cards can then be used independently as a pack of cards by individual children, as they are self-correcting.

If you have not tried this before, I recommend starting with one or two examples with the whole class together, and then once they are used to the technique, try arrangement (e). But please, bear in mind that nobody knows your class like you do - there are plenty of options! Pay particular attention to the part where EVERYONE must agree. This is where the ASN strategy has the potential to benefit all five of our fictitious groups.

Key Stage 1 Example

This card shows a possible sample set for children who have been thinking about odd and even numbers. I chose this one as there are 2 statements for ‘Always’, 2 for ‘Sometimes’ and 2 for ‘Never’.

Watching and listening to children work through this task will give you a vast amount of insight into their misconceptions, in a way that few other tasks can match.

Key Stage 2 Example

You can see here that this set of questions is designed to specifically target misconceptions; triangles can NEVER have exactly 2 lines of symmetry, though they may have zero, one or three.

Also, squares ARE rectangles! Not understanding this is one of the most common misconceptions I have come across in primary mathematics.

Again, watching children as they discuss and seek to convince each other is a fascinating and informative process for the teacher - a wonderful chance to listen rather than speak.

Personal note: *This task was popularised, and very possibly invented, by the late Malcolm Swan of Nottingham University. I was so struck by its effectiveness that I wrote a book of starter ASN tasks, from which the two examples above are taken. It is still available from andrewjeffrey.co.uk. While having dinner with Malcolm in Stirling in 2016 we decided to write a sequel together, which we were going to call 'Always, Sometimes, Never Again'. Sadly Malcolm passed away in 2017 and the book was never written. Malcolm's contribution to the maths education community was immense, and he was taken from us too soon. He is sorely missed*.

5. 'How' not 'What'

In a Nutshell: Give a problem, but don't ask for the answer. Ask instead HOW the children think they could solve it.

This is one of my favourite teaching tools. It is potentially effective for those children in group 1 and 2, who are not sure how to tackle a problem, and therefore as we saw earlier, simply cannot engage fruitfully with 'answer-focused' questioning.

It is necessary to be fairly strict and pedantic when using this technique, and be prepared to ask supplementary questions based on the children's responses. Here is how it works:

Instead of asking this:
"*I run 2 miles a day. How far do I run in a week?*"

We actually ask this:
"*I run 2 miles a day. How can I work out how far I run in a week?*"

The correct answer to the first question is of course 14 miles, but it is NOT the correct answer to what we actually asked. The correct answer is of course "Multiply 2 by 7." Better still, as we saw in the introduction to this section, would be "*Multiply 2 by 7 <u>because</u> there are seven days in a week so we need to find seven 2s."*

As you collect answers, make a written or drawn note of all methods offered. It is VITAL that you do not, either in any obvious or subtle way, appear to overly validate, dismiss or disapprove of *any of the suggestions*, whether or not they are correct or efficient. We are all guilty from time to time of this silent signalling of our disapproval- the raised eyebrows, the slight frown, the “Are you sure?’ response; these are a complete no-no when it comes to motivating the less confident children who actually risk making a suggestion. It is almost always the case that children naturally ‘peer review’ the methods that are offered by others, and while this must be carefully and sensitively managed, it is perfectly possible to have a conversation about ideas rather than the people who offered them, particularly if the original suggester was praised specifically for sharing their thoughts.

This technique has four key benefits.

Children get to listen to a range of approaches that may

a) help them understand a problem that they had no way into previously
b) make them realise that their own way may not be the only way, or even the most efficient
c) hear it explained by their peers using their own language rather than ‘teacher-speak’, and
d) Allow them to change their mind about their preferred approach, without having to publicly admit their original way was wrong or less efficient and thus avoid saving face.

Key Stage 1 Example:

“*How could we work out 19 plus 19?*”

You will almost certainly receive a range of answers including, but not limited to such things as: '*column addition*', '*add the tens first'*, '*add the ones first*' '*make it 20 plus 20 and then take away*', etc.

You will also get incorrect methods offered, and as mentioned previously it is important to work through those and show that they do not give us the correct answer, but perhaps they could be adapted? Work the question out using each offering. Yes, this is an investment of time but there are very few more sound investments at this stage. Rest assured that this one will pay massive dividends.

Other things you might ask once children have gone through this, might include "*What methods can you find?*" "*What do you think the easiest way is?*" "*Which is your favourite - can you explain why you like it?*"

Key Stage 2 Example

"*How would you work out 20% of £35.00 ?*"

You can probably anticipate many of the answers. Some likely candidates are:

"*Divide £35 by 5 as 20% is one fifth*"
"*Find 10% then double*"
"*1% of £35 is 35p so multiply 35p by 20*"

Again, taking the time to consider all of these in front of the whole class will help children build a greater knowledge and deeper understanding of how percentages operate; they are just fractions and operate in exactly the same way. By doing all of these there is a high chance that at least one of them will resonate with a child who is unsure how to proceed.

A final tip, (admittedly more to do with percentages than the ‘how’ technique) is to use plenty of drawings as you work through children’s suggested methods. These can engage children in a different way, rather than them seeing just numbers, and help them make connections between percentages and other fractions - even more strongly so if they have studied bar modelling.

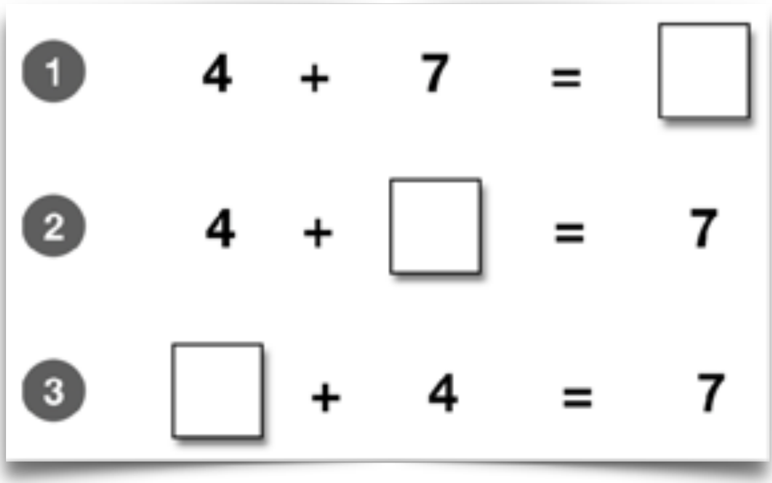
6. What's Missing?

In a Nutshell: Give a calculation with the answer included, but with some digits missing in the question, and ask children to work out what the missing digits could be.

There is almost limitless potential with this technique. It does require an accurate understanding of the equals sign as we will see, but as a way of encouraging deeper algebraic thinking it is very effective. Very popular with National Curriculum test setters, the strength of this tool is to help children focus on the relationship between inverse operations.

I am not a huge fan of calling them 'inverse operations' but I completely understand why the nomenclature has gained traction. The implication is that addition and subtraction are opposites, as are multiplication and division, but the relationship is deeper than that - it can be argued for example that addition and subtraction are the same thing! Before you throw this book across the room, let me explain. They are the same in as much as they seek to quantify the relationship between two or more parts and their total. If it is the total we wish to find, given the parts, we call this addition. But if we know the total already, and it is one of the parts that we are required to find, we call this subtraction.

In the same way, although multiplication may be thought of in many ways, if we thing about it as 'lots of' something, then division simply seeks to find how many 'lots of' there are.



We could discuss this further, but hopefully this is sufficient to suggest that we should practice addition and subtraction together, and the same is true for multiplication and division.

Let's look at a few different examples.

Key Stage 1 Examples

Perhaps the simplest of all examples would be something like this. Notice the progression in complexity as the questions progress.

Example 1 requires children to perform an addition and write the answer in the box.

Example 2 however has a higher cognitive expectation. Many children who first come across examples of this type write 11 in the box. This is not surprising, but it is wrong(!) and avoidable.

Example 3 feels harder still. In the first two examples, a child can use a number line either on paper or in their head and start at 4. In example 3 however no such starting point is provided and the the child will need to do a further piece of reasoning, either by counting back from 7 or using their knowledge of commutativity.

One key to success here is the use of a pan balance. If you have Numicon shapes, the fact that their mass is proportional to their size (for example a 3-shape and a 2-shape will balance with a 5-shape) is invaluable. Putting a 4 in one side, a 7 in the other, and asking "*What shape needs to go in with the 4 to make the scales balance?"* is a perfect way to bridge to the abstract image shown in questions 2 and 3 above.

Redrawing equals signs as a seesaw can also very effective:

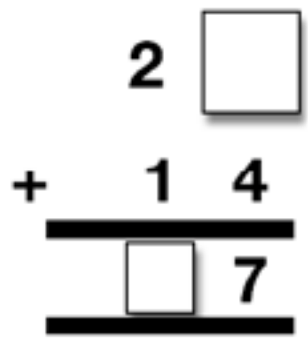
This visual image has proved powerful over the years with KS1 children who are not yet convinced that 7 = 3 + 4, despite being happy that 3 + 4 = 7. This is due to a misunderstanding of the equals sign to mean '*makes*' or 'is *identical to*' or '*and now here comes the answer*'. Modelling with a pan balance, and rotating the balance in front of them almost always solves this problem.

In this image, the teacher has stuck two pieces of paper with lines to the base and the arm - they will read '=' only when the pan is balanced.

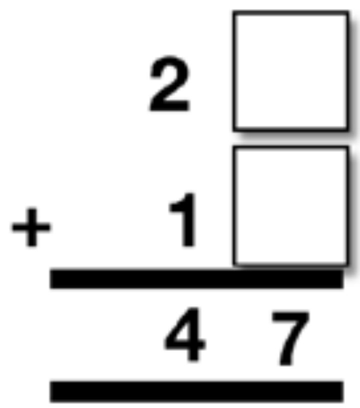
It is also worth remembering the part-whole model discussed in the section on ‘Making Connections’. It is an ideal precursor to the empty-box type of question.

Key Stage 2 Examples

A good way to use the empty box concept is in the formal layout of calculations. Here is an example which requires greater depth of thinking than merely asking children to add 24 and 14 together: The thought process needs to be: *something plus 4 makes seven…this is not intuitive since 7 is greater than 4…this pushes children to realise that the 4 must be part of 14…and so on. Greater Depth*.



The example above has only one correct solution, but we can make it an even richer task by offering questions with more than one solution. For example, this question will keep children fruitfully occupied for longer than the first. Although similar, there is more than one solution. Asking children to find a solution, then compare solutions, then find all possible solutions, is an easy way to enable them to think more deeply.



Another terrific time-saving tool when designing ‘what’s missing’ questions is to use a grid such as the one on the following page.

Year	Standard	Deeper
Y1	11 + 6 = ?	11 + ? = 17
Y2	35 + 37 = ?	? + 37 = 72
Y3	24 x 3 = ?	? x 3 = 72
Y4	4782 - 635 = ?	4782 - ? = 4147
Y5	5/6 x 8 = ?	5/6 x ? = 40/8
Y6	11/4 - 4/3 = ?	11/4 - ? = 17/12

In the ‘Standard’ column, write down some calculations appropriate for the year group. In the ‘Deeper’ column, simply rewrite the question but with the answer included and a bit of the question missing.

Finally, as children progress through Key Stage 2, it is a neat idea to start writing an ‘x’ on the box, meaning they are now beginning to solve things such as this.

$$\boxed{x} - 7 = 4$$

7. C.P.A.

In a Nutshell: Use concrete resources and imagery to scaffold children's thinking, so that they are capable of more abstract thought.

Whole books could (and have) been written about this. As a good example, see Griffiths, Back and Gifford's excellent 'Making Numbers', Oxford, (2019). However, in this book we will limit ourselves to a brief look at the 'how' and 'why' of the CPA approach. This strategy is terrific for under-confident children in the class. but also for anyone starting a new concept. The ability to model and physically manipulate scenarios using equipment is invaluable for children who are making sense of such an abstract thing as mathematics.

Briefly: imagine taking a set of three objects and adding a fourth. This is 'doing' mathematics, and is what Jerome Bruner referred to as the '*enactive*' phase. Creating some sort of pictorial representation of this action, perhaps with four dots and an arrow, is what he called the '*iconic*' phase.

Finally, being able to record and understand this as "3+1=4" is known as the '*symbolic'* phase; the symbols do not look like the actual thing that took place; in fact they are a social agreement for symbols on which be have bestowed meaning. But meaning takes time to develop, and so we must not be surprised if children are not able to perform what we would consider relatively simple calculations.

Until children have a fluent recall and strong concept image of numbers and the symbols we use to represent them, they will inevitably be held back. For this reason, it is beholden upon us to make available such practical equipment as might help children form mental images of the mathematical ideas we are asking them to consider.
Not only should we make it available, but we must also normalise its use to avoid stigmatising any child who is able to think abstractly more slowly than their peers. It is impossible to produce a definitive list, and needs differ from class to class, but in general I would recommend the following be made freely available:

> *Counters, dice, coins, bead-strings, Numicon shapes, playing cards, tens frames, straws or pipe-cleaners, geoboards and rubber bands, 2d and 3d shapes, Dienes blocks, rulers, pattern blocks, place value cards, place value counters, fraction cards, linking cubes, strips of paper, protractors, pairs of compasses, digit cards or counters, spinners, Cuisenaire rods and number tracks.*

There will of course be other resources that teachers are familiar with, so this is not intended by any means to be an exhaustive list. The crucial thing with any equipment though is to think carefully how and why it will be used. The 'why' is simple enough; to serve as a metaphor for the abstract, and as such it is important not to let children become too dependent on the resources.

The 'how' is more of a challenge, but there is a useful rule of thumb. When you are teaching the start of a new topic, try to plan ways to do so using the concrete resources. This gives all children a foundation on which to build visual and then abstract thinking. They will move away from them at different speeds, and this does not matter a jot. What does matter is the classroom culture that says the use of resources is not inferior. For this reason I like to keep a box of commonly used equipment handy in the centre of the tables, just like whiteboards, pens, etc. This sends the signal that it is fine and normal to use them whenever required and normalises working practically. We must remove the stigma of working in a practical way in mathematics at all costs if we are go give children the best chance to understand more deeply what they are doing.

Final thought - the move through concrete, pictorial and abstract is not intended to be a one-way street. It may often be necessary and even desirable to move back to a concrete example, or to draw a picture even when the concept is understood. The bar model is a good example of this, as is the use of geoboards when thinking about area.

Key Stage 1 Example

There are a myriad of possibilities of course, so for the sake of time we will just go for a classic:

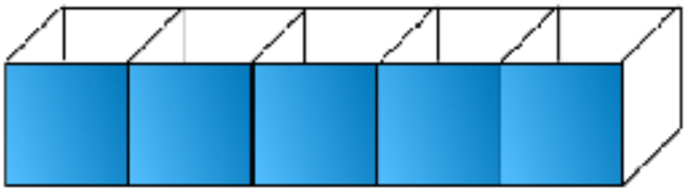
"*What happens when we add two odd numbers together?***"**

This is an abstract thing to ask, so using tens frames or Numicon shapes will help children get a sense of the two 'odd bits' coming together to make a pair, and hence they will be more easily able to generalise that the sum will always be even.

Key Stage 2 Example

"*Three fifths of my number is 12. What is my number?*"

This is a difficult thing to think about - more typically children are given a number and asked to find three fifths of it. How might this be scaffolded? Linking cubes are a great tool for this - invite them to make a stick from cubes that can be cut into fifths. The simplest way to do this of course is with 5 cubes. I like to ask children to imagine they are containers:



Now draw a simplified version on the board (or ask them to). Highlighting 3 of the cubes and assigning a total value of 12 to them makes it easier to see that each cube must be worth 12÷3. This is 4, so the whole bar is worth 4 x 5 or 20.

8 Which is hardest?

In a Nutshell: Give children three examples of a calculation, for example three different multiplications of two-digit numbers. Ask the class to rank the examples in what they believe to be the order of difficulty.

This is not to be confused with the popular technique of assigning work at three levels of difficulty and calling them, for example, **mild spicy** and **hot** (*we seem to like using chilli analogies in the UK*). This is good practice too, though to maximise its effectiveness does take some thought - children should always be given examples of each type before being asked to choose, as this increases the likelihood that is they will choose appropriately.

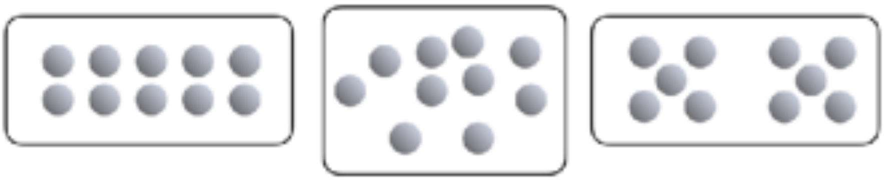
There are two main ways to implement this strategy. The first is to show children three questions, ask them to work through them and ask them to rank them in order of complexity. This of course involves significant metacognition, one of our four characteristics for encouraging greater depth (see part one).

The second way, which will take more time to get right, is to ask children themselves to create easy, moderate and tricky versions of a question of their own choosing, on a topic they have been studying. This is best used summatively at the end of a unit of work, once children appear to have mastered the content.

Key Stage 1 Example

“Look at these three sets of objects. Which do you think would be easiest and hardest to count. Why?”

Counting dots is an extremely rich task in itself, as it allows children to practice subitising (recognising small quantities without having to count) and pattern recognition. Notice that in our example each box has ten dots, so in order to ascribe relative difficulties we are forcing children to think more deeply about the underlying structure of the dots rather than merely the quantities; it is this awareness of the structures of mathematics that will lead to the deep thinking we all desire.



Key Stage 2 Example

I like to show these three calculations to parents whenever I am invited to speak at school information evenings, and ask them to rank them in order of difficulty.

A

$$\begin{array}{r} 4328 \\ +\ 5371 \\ \hline \\ \hline \end{array}$$

B

$$\begin{array}{r} 124 \\ +\ 857 \\ \hline \\ \hline \end{array}$$

C

$$\begin{array}{r} 24 \\ +\ 77 \\ \hline \\ \hline \end{array}$$

It takes a while for the penny to drop, but eventually they realise that A is easiest, followed by B, with C being the most difficult. This is counter-intuitive, and forces them away from the ‘bigger numbers are harder' narrative; this can be very compelling and most importantly, help them see the underlying structures of the column addition layout. It is not limited to calculations - missing angle problems, area calculations, even missing number problems (technique 6) all work extremely well to this idea.

9. Who Heard?

In a Nutshell: Ask a question. Give children time to work on the answer individually, then ask them to explain the answer they have worked out to a partner. Finally, instead of asking children for their *own* explanation, ask instead them to raise a hand if they *heard* a good explanation from their partner, and then get them to repeat it in their own words.

This technique works best as a follow-on to ‘think pair share’. In fact it can be thought of as an extension of that idea, but with an important difference, specifically that my idea can only be shared publicly by my partner and not by me, thus encouraging me to think deeply about how I explain it to my partner.

I developed this technique by thinking about how best to engage as many of the five groups as possible. There are often shy children in group 3 who might never dare to raise a hand and offer their insights to an entire class, but who might just be prepared to do so with a trusted partner. This technique therefore affords them a chance to have their thoughts heard, but with little person risk. Children in groups 1 and 2 also benefit, as long as they are paired with someone who has some insight. As a added bonus they have a chance to be praised, simply for listening to someone else’s explanation. This works because once you ask the ‘*Who heard a good explanation?*’ question, your first job is to praise the owner of the hand that goes up- for good listening.

Earlier we met Daniel. He benefited hugely from ‘Who Heard?’, as he was now forced to explain his ideas to another person in such a way that they in turn would feel confident to explain them to the rest of the class.

By now you will probably have realised how important it is to get pairings correct. Pair shy children with patient confident ones where you can, and pair effusive children with good listeners if possible.

In a class you do not know this can be tricky so trial and error on the teachers part will be necessary in the initial stages. I once found myself teaching maths in Thailand to a Year 4 class I had not previously met. One enthusiastic young man was excitedly calling out every time I asked a question, so I decided to employ the 'Who heard?' Technique to allow other children to get a word in. Obviously not knowing the class it made sense for the class teacher to select the pairings herself.

I asked the question and explained how we would proceed. I listened to the class discussions, and overheard the boy say to his partner "*OK, listen carefully. I'm going to tell you the answer and then you have to put your hand up and say you heard a good explanation.*" His quest for validation knew no limits!

Key Stage 1 Example

"How could we find a quarter of these sweets"?

You need to employ good listening skills as you go round the class. Sometimes children need a nudge, but it is worth being prepared to ask a few different people what they heard. If they are able to articulate it, praise the listening and then praise their partner for providing the original explanation.

Key Stage 2 Example

“Why are square numbers called square numbers?”

Sometimes you will get no takers. In this case, simply say “*Then your explanations need to be better - try again*.” Stick to your guns, here, as this encourages deeper thinking!

10. What if?

In a Nutshell: At any stage of working on a problem, ask children to think about an aspect of the maths that they could change to improve or make the problem, easier, harder, or just different.

For our final strategy, I have attempted to bring the book full circle, by finishing where we started - with curiosity. The two words ‘What If’ are amongst my absolute favourites in a mathematics classroom. This is exactly the type of thinking question that helps children along the journey of curiosity, where we quite rightly began this book.

The scope for this strategy is virtually boundless. Consider these examples, which in this case we have not bothered to split into key stages as most work equally well in either KS1 or KS2.

- *What could you change about this to make it easier/harder?*
- *What new rule could you add to make this game even more exciting?*
- *What if we did…instead of…?*
- *What would happen if all the numbers in the calculation were doubled/halved/increased by 1?*
- *What if we kept the perimeter the same - how could we change the area?*
- *What if we put the numbers the other way round?*
- *What if we make them all ten times bigger?*
- *What could we change/wonder about this question?*
- *What if you were only allowed to use odd numbers?*
- *Would it still be possible if…?*

You will think of many more, but the essence of all of these is to spark curiosity - and Greater Depth.

And Finally

If you refer back to the beginning of section 2, you will recall our five groups of children. All of the techniques we have considered are designed to afford opportunities to each group to develop deeper mathematical thinking, and introduce them to the wonder of wondering.

It probably goes without saying that not all of them will work with every group of children. Some will work, with some children, some of the time.

Feel free to try, experiment, and above all do whatever you can to encourage ALL your children to enjoy learning mathematics; it will be one of the greatest gifts you can bestow upon them.

A few important words of thanks: Jonathan Bull, my inspirational boyhood maths teacher; Lucy Rycroft-Smith for pointing out how awful my original cover was and designing a much cooler one; Lynne McClure, my mentor and friend for many years whose wisdom I treasure and whose edits of a previous draft were a huge improvement; Tony Wing, my friend, co-author, and my university tutor, and the person who made me think in greater depth about maths education than I have ever done before; Alison, my wonderful wife, for tolerating my mathematical obsessions with good grace; and to the wonderful people in the mathematics community who have been unfailingly encouraging and kind - I value you all.

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